

Fermion to boson mappings revisited

Joseph N. Ginocchio and Calvin W. Johnson

T-5, MS B283, Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545

Abstract

We briefly review various mappings of fermion pairs to bosons, including those based on mapping operators, such as Belyaev-Zelevinskii, and those on mapping states, such as Marumori; in particular we consider the work of Otsuka-Arima-Iachello, aimed at deriving the Interacting Boson Model. We then give a rigorous and unified description of state-mapping procedures which allows one to systematically go beyond Otsuka-Arima-Iachello and related approaches, along with several exact results.

I. INTRODUCTION

Professor Belyaev (with V. G. Zelevinskii) (BZ) pioneered the mapping of fermion systems onto bosons more than thirty years ago [1]. These original attempts at bosonization of fermion systems was motivated by collective particle-hole modes in nuclei. Since that time the interacting boson model [2] (IBM) has been phenomenologically very successful in explaining low energy nuclear spectroscopy for heavy nuclei. The bosons in this model are thought to represent monopole ($J=0$), quadrupole ($J=2$), and sometimes hexadecapole ($J=4$) correlated pairs of valence nucleons in the shell model. The IBM Hamiltonian is Hermitian, usually has at most two-boson interactions, and conserves boson number, reflecting the particle-particle, rather than particle-hole, nature of the underlying fermion pairs. While one can numerically diagonalize the general IBM Hamiltonian, one of the strengths of IBM are the algebraic limits corresponding to the subgroups $SU(3)$, $U(5)$, or $O(6)$, with analytic

expressions for excitation bands and transition strengths, which encompass an enormous amount of nuclear data.

The microscopic reasons for the success of such a simple model are elusive. Otsuka, Arima, and Iachello, along with Talmi, have used a mapping of the shell model Hamiltonian to the IBM Hamiltonian [3,4] based on the seniority model [5], but these attempts have not done well for well-deformed nuclei. For this reason we have revisited boson mappings to see if we can understand the success of the IBM starting from the shell model.

In the next section we sketch out various historic approaches to boson mappings [6]. We follow Marumori [7] and Otsuka et al. [3,4] (OAI) in our mapping procedure which maps fermion states into boson states and construct boson operators that reproduce fermion matrix elements. We give the boson representation of the Hamiltonian and review the result that, in the full space, it factorizes into a boson image, which is the same as the BZ Hamiltonian, times a normalization operator which projects out the spurious states. However, since our goal is to understand the IBM which only deals with a few of the enormous degrees of freedom of the shell model, we go on to discuss boson images in truncated spaces. This, we shall see, gives rigorous insight into the OAI mapping and shows how to go systematically beyond it.

II. A BRIEF HISTORY OF BOSON MAPPINGS

The fundamental goal is to solve the many-fermion Schrödinger equation

$$\hat{H} |\Psi_\lambda\rangle = E_\lambda |\Psi_\lambda\rangle \quad (1)$$

and find transition matrix elements between eigenstates, $t_{\lambda\lambda'} = \langle \Psi_\lambda | \hat{T} | \Psi_{\lambda'} \rangle$. As the fermion Fock space may be so large as to make direct solution intractable, the idea of a boson mapping is to replace the fermion operators with boson operators, *using only a minimal number of boson degrees of freedom*, that approximate the spectrum and transition matrix elements of the original fermion problem. There are two approaches to boson mappings which we now review.

The first approach, epitomized by Belyaev and Zelevinskii [1], is to map fermion operators to boson operators so as to preserve the original algebra. Specifically, consider a space with 2Ω single-fermion states; a_i^\dagger, a_j signify fermion creation and annihilation operators. The set of all bilinear fermion operators, $a_i a_j, a_k^\dagger a_l^\dagger, a_i^\dagger a_j$, form the Lie algebra of $\text{SO}(4\Omega)$, as embodied by the commutation relations

$$[a_i a_j, a_k a_l] = 0 \quad (2)$$

$$[a_i a_j, a_k^\dagger a_l^\dagger] = \delta_{il} \delta_{jk} + \delta_{ik} a_l^\dagger a_j + \delta_{jl} a_k^\dagger a_i - (i \leftrightarrow j) \quad (3)$$

$$[a_i a_j, a_k^\dagger a_l] = \delta_{jk} a_i a_l - (i \leftrightarrow j) \quad (4)$$

$$[a_i^\dagger a_j, a_k^\dagger a_l] = \delta_{jk} a_i^\dagger a_l - \delta_{il} a_k^\dagger a_j \quad (5)$$

At this point it is convenient to introduce collective fermion pair operators

$$\hat{A}_\beta^\dagger \equiv \frac{1}{\sqrt{2}} \sum_{ij} (\mathbf{A}_\beta^\dagger)_{ij} a_i^\dagger a_j^\dagger. \quad (6)$$

We always choose the $\Omega(2\Omega - 1)$ matrices \mathbf{A}_β to be antisymmetric to preserve the underlying fermion statistics, thus eliminating the need later on to distinguish between ‘ideal’ and ‘physical’ bosons. We also assume the following normalization and completeness relations for the matrices:

$$\text{tr } \mathbf{A}_\alpha \mathbf{A}_\beta^\dagger = \delta_{\alpha\beta}; \quad (7)$$

$$\sum_\alpha (\mathbf{A}_\alpha^\dagger)_{ij} (\mathbf{A}_\alpha)_{j'i'} = \frac{1}{2} (\delta_{ii'} \delta_{jj'} - \delta_{ij'} \delta_{ji'}). \quad (8)$$

Generic one- and two-body fermion operators we represent by $\hat{T} \equiv \sum_{ij} T_{ij} a_i^\dagger a_j$, $\hat{V} \equiv \sum_{\mu\nu} \langle \mu | V | \nu \rangle \hat{A}_\mu^\dagger \hat{A}_\nu$, where $T_{ij} = \langle i | \hat{T} | j \rangle$; from such operators one can construct a fermion Hamiltonian \hat{H} . Now one has the following commutation relations:

$$[\hat{A}_\alpha, \hat{A}_\beta] = [\hat{A}_\alpha^\dagger, \hat{A}_\beta^\dagger] = 0; \quad (9)$$

$$[\hat{A}_\alpha, \hat{A}_\beta^\dagger] = \delta_{\alpha\beta} - 2 \sum_{ij} (\mathbf{A}_\beta^\dagger \mathbf{A}_\alpha)_{ij} a_i^\dagger a_j; \quad (10)$$

$$[\hat{A}_\alpha, \hat{T}] = 2 \sum_\beta \text{tr } (\mathbf{A}_\alpha \mathbf{T} \mathbf{A}_\beta^\dagger) \hat{A}_\beta \quad (11)$$

$$[\hat{T}_1, \hat{T}_2] = \sum_{ij} (\mathbf{T}_1 \mathbf{T}_2 - \mathbf{T}_2 \mathbf{T}_1)_{ij} a_i^\dagger a_j \quad (12)$$

The method of Belyaev and Zelevinskii is to find boson images of the bifermion operators,

$$(\hat{A}_\mu^\dagger)_B = b_\mu^\dagger + \sum_{\alpha\beta\gamma} x_\mu^{\alpha\beta\gamma} b_\alpha^\dagger b_\beta^\dagger b_\gamma + \sum_{\alpha\beta\gamma\delta\epsilon} x_\mu^{\alpha\beta\gamma\delta\epsilon} b_\alpha^\dagger b_\beta^\dagger b_\gamma^\dagger b_\delta b_\epsilon + \dots \quad (13)$$

$$(\hat{A}_\mu)_B = (A_\mu^\dagger)_B^\dagger \quad (14)$$

$$(\hat{T})_B = \sum_{\alpha\beta} y^{\alpha\beta} b_\alpha^\dagger b_\beta + \sum_{\alpha\beta\gamma\delta} y^{\alpha\beta\gamma\delta} b_\alpha^\dagger b_\gamma^\dagger b_\beta b_\delta + \dots \quad (15)$$

where $b_\alpha, b_\beta^\dagger$ are boson creation and annihilation operators, $[b_\alpha, b_\beta^\dagger] = \delta_{\alpha\beta}$, with the coefficients x, y chosen so that the images $(A_\mu^\dagger)_B, (A_\nu)_B, (T)_B$ have the same commutation relations as in (9)-(12). Because the algebra is exactly matched, if one builds boson states in exact analogy to the fermion states then the full boson Fock space is not spanned and one does not have nonphysical or spurious states. For the state-mapping methods described below, especially when the entire boson Fock space is used, identifying and decoupling from spurious states is an important and problematic issue.

On the other hand, the Belyaev-Zelevinskii expansion is in general infinite. In the full boson Fock space, that is, no truncation of the boson degrees of freedom, the image of one body operators is finite and given quite simply by $(\hat{T})_B = 2 \sum_{\alpha\beta} \text{tr}(\mathbf{A}_\alpha \mathbf{T} \mathbf{A}_\beta^\dagger) b_\alpha^\dagger b_\beta$. Since any fermion Hamiltonian can be written in terms of one-body operators, the boson image of a finite fermion Hamiltonian can be finite in the full space. The states that one must use then are built from the pairs given in (13) which will not just be products of bosons but will include exchange terms. For example, for two bosons and using (13),

$$\hat{A}_\alpha^\dagger \hat{A}_\beta^\dagger |0\rangle \rightarrow (b_\alpha^\dagger b_\beta^\dagger + x_\alpha^{\sigma\tau\beta} b_\sigma^\dagger b_\tau^\dagger) |0\rangle. \quad (16)$$

These exchange terms are due to the antisymmetry. We shall take care of such exchange effects by introducing a norm operator in the boson space. For truncated spaces, however, the expansion of the BZ Hamiltonian is infinite.

The Dyson mapping [8] is a variant of Belyaev-Zelevinskii, in which one makes the mapping

$$\hat{A}_\alpha \rightarrow b_\alpha; \quad (17)$$

$$\hat{A}_\beta^\dagger \rightarrow b_\beta^\dagger - 2 \sum_{\lambda\mu\nu} \text{tr}(\mathbf{A}_\lambda \mathbf{A}_\beta^\dagger \mathbf{A}_\mu \mathbf{A}_\nu^\dagger) b_\lambda^\dagger b_\mu^\dagger b_\nu \quad (18)$$

$$\hat{T} \rightarrow \sum_{\alpha\beta} 2 \text{tr}(\mathbf{A}_\alpha \mathbf{T} \mathbf{A}_\beta^\dagger) b_\alpha^\dagger b_\beta. \quad (19)$$

$$(20)$$

The operators are then clearly finite; on the other hand they are just as clearly non-Hermitian. From a computational viewpoint non-Hermiticity is only a minor barrier, but it is an obstacle to an understanding of the microscopic origin of Hermitian IBM Hamiltonians. Furthermore the Dyson operators mix spurious and physical spaces.

Marshalek [9] points out there exist mappings that are both finite and Hermitian, but these in general require projection operators to eliminate spurious states. We will regain this result later on in this paper.

The second major approach, pioneered by Marumori [7], is to map fermion states and construct boson operators that preserve matrix elements. For fermion many-body (shell-model) basis states one often uses Slater determinants, antisymmetrized products of single-fermion wavefunctions which we can write using Fock creation operators: $a_{i_1}^\dagger \cdots a_{i_n}^\dagger |0\rangle$ for n fermions. For an even number of fermions one instead constructs states from products of fermion pairs,

$$|\Psi_\beta\rangle = \prod_{m=1}^N \hat{A}_{\beta_m}^\dagger |0\rangle; \quad (21)$$

if the number of fermion is fixed at n then m runs from 1 to $N = n/2$. The original work of Marumori, however, focused on particle-hole excitations and so the number of pairs and consequently bosons is not fixed. These states are not trivially orthonormal. They must be orthogonalized. Exactly how this orthogonalization is accomplished will be a key theme in this paper.

Marumori constructs the norm matrix

$$\mathcal{N}_{\alpha\beta} = \langle \Psi_\alpha | \Psi_\beta \rangle \quad (22)$$

and then the Usui operator

$$U = \sum_{\alpha, \beta; n} |\Phi_\beta\rangle (\mathcal{N})_{\beta\alpha}^{-1/2} \langle \Psi_\alpha| \quad (23)$$

where bosons states are constructed in strict analogy to the fermion states,

$$|\Phi\rangle = \prod_{m=1}^N b_{\beta_m} |0\rangle. \quad (24)$$

Then the Marumori expansion of any fermion operator is

$$O_B = U O_F U^\dagger. \quad (25)$$

Clearly Marumori is best suited for particle-hole states with only a few excitations. If one applied it to a system with numerous particle-particle pairs, as for the IBM, one obtains clumsy many-body terms. Kishimoto and Tamura [10] addressed this last issued by introducing a “linked-cluster” expansion which they then grafted into a BZ-type scheme.

Otsuka, Arima, and Iachello (OAI), along with Talmi [3,4], investigated the microscopic origins of the Interacting Boson Model through boson mappings. Although they also mapped states, they differed from Marumori in some key details. First of all, they built states built on a fixed number of particle-particle, not particle-hole, pairs, and restricted the pairs to one monopole ($J^\pi = 0^+$) and quadrupole ($J^\pi = 2^+$) pair. These states were orthogonalized based on seniority. That is, they construct, for $2N$ fermions, low-seniority basis states of S and D fermion pairs, $|S^{N-n_d} D^{n_d}\rangle$, and then orthonormalize the states such that the zero-seniority state is mapped to itself, and states of higher seniority v are orthogonalized against states of lower seniority,

$$|v\rangle \rightarrow |“v”\rangle = |v\rangle + |v-2\rangle + |v-4\rangle + \dots \quad (26)$$

Then OAI calculate the matrix elements $\langle “S^{N-n'_d} D^{n'_d}” | H_F | “S^{N-n_d} D^{n_d}” \rangle$ for $n_d, n'_d = 0, 1, 2$ and obtain the coefficients for their one plus two-boson Hamiltonian. These coefficients have an explicit N -dependence (and for large N and arbitrary systems such matrix elements are not trivial to calculate, especially in analytic form!) and thus implicitly a many-body dependence. At first sight this is not entirely unreasonable as it is well known the IBM parameters change substantially as a function of the number of bosons, even within a major

‘shell’. Nonetheless the OAI mapping has three drawbacks. The first is that it’s not clear how to systematically calculate many-body contributions beyond that contained in the OAI prescription, whereas the method we shall describe is fully and rigorously systematic. The second is that the OAI prescription can induce many-body effects where none are needed. This point will be illustrated in section IV C. Thirdly, only the $n_d = 0, 1, 2$ space is exactly mapped, but very deformed systems will involve large n_d . In fact, for an axial rotor limit, the average number of d-bosons in the ground state band is $2/3$ the total number of bosons.

As an alternative to OAI, Skouras, van Isacker, and Nagarajan [11] proposed a “democratic” mapping where the orthogonalization is based on eigenvectors of the norm matrix rather than seniority.

In what follows we attempt to rigorously unify all the state-mapping methods. We have three strong results. First, we give general expressions for fermion matrix elements via boson representations. Second, we show how in several cases one can have exact, finite, and Hermitian boson images of fermion operators. Finally, we show how to extend both the OAI and democratic mappings in a systematic and rigorous fashion, and illustrate how the choice of orthogonalization can affect the many-body dependence of the boson images.

III. BOSON REPRESENTATIONS OF FERMION MATRIX ELEMENTS

The starting point of any state-mapping method is the calculation of matrix elements of fermion operators between states constructed from fermion pairs of the form (21), including the overlap: $\langle \Psi_\alpha | \Psi_\beta \rangle$, $\langle \Psi_\alpha | \hat{H} | \Psi_\beta \rangle$, $\langle \Psi_\alpha | \hat{T} | \Psi_\beta \rangle$, and so on. These matrix elements are much more difficult to compute than the corresponding matrix elements between Slater determinants. As we shall show, however, full and careful attention paid to the problem of calculation matrix elements can yield powerful results. Silvestre-Brac and Piepenbring [12], laboriously using commutation relations, derived a Wick theorem for fermion pairs. Rowe, Song and Chen [13] using ‘vector coherent states’ (we would say fermion-pair coherent states) found matrix elements between pair-condensate wavefunctions, states of the form $(\hat{A}^\dagger)^N |0\rangle$.

Using a theorem by Lang et al. [14], we have generalized [15] the method of Rowe, Song and Chen and recovered (actually discovered independently) the expressions of Silvestre-Brac and Piepenbring. Specifically, we construct generating functionals by taking the matrix element

$$\begin{aligned} & \left\langle 0 \left| \exp \left(\sum_{\alpha} \epsilon_{\alpha} \hat{A}_{\alpha} \right) \exp \left(\sum_{\beta} \epsilon_{\beta} \hat{A}_{\beta}^{\dagger} \right) \right| 0 \right\rangle \\ &= \exp \left(\sum_{k=1}^{\infty} \frac{(-2)^{k-1}}{k} \text{tr} \left[\sum_{\alpha\beta} \epsilon_{\alpha} \epsilon_{\beta} \mathbf{A}_{\alpha} \mathbf{A}_{\beta}^{\dagger} \right]^k \right). \end{aligned} \quad (27)$$

By taking derivatives of ϵ_{α} , etc., one computes the desired matrix elements in analytic form [15]. For pair condensate wave functions one can calculate the matrix elements iteratively and propose a variational principle [13]. Such a variational principle would be useful in determining the “best” microscopic structure for a truncated set of pairs. Alternately, Otsuka and Yoshinaga [16] choose their S and D pairs from Hartree-Fock-Bogoliubov states; the two approaches can probably be related in some approximation. This may be important in answering a basic question of IBM, the origin of algebraic limits: do they arise from changes in pair structure, or from effective many-body effects, or both?

We now want to translate the fermion matrix elements into boson space. We take the simple mapping of fermion states into boson states

$$|\Psi_{\beta}\rangle \rightarrow |\Phi_{\beta}\rangle = \prod_{m=1}^N b_{\beta_m}^{\dagger} |0\rangle, \quad (28)$$

where the b^{\dagger} are boson creation operators. We construct boson operators that preserve matrix elements, introducing boson operators $\hat{\mathcal{T}}_B$, $\hat{\mathcal{V}}_B$, and most importantly the *norm operator* $\hat{\mathcal{N}}_B$ such that $(\Phi_{\alpha} | \hat{\mathcal{T}}_B | \Phi_{\beta}) = \langle \Psi_{\alpha} | \hat{T} | \Psi_{\beta} \rangle$, $(\Phi_{\alpha} | \hat{\mathcal{V}}_B | \Phi_{\beta}) = \langle \Psi_{\alpha} | \hat{V} | \Psi_{\beta} \rangle$. and $(\Phi_{\alpha} | \hat{\mathcal{N}}_B | \Phi_{\beta}) = \langle \Psi_{\alpha} | \Psi_{\beta} \rangle$. We term $\hat{\mathcal{T}}_B, \hat{\mathcal{V}}_B$ the boson *representations* of the fermion operators \hat{T}, \hat{V} . One finds the ‘linked-cluster’ (a la Kishimoto and Tamura [10] although with differences) expansion of the representations to be of the form [15]

$$\hat{\mathcal{N}}_B = 1 + \sum_{\ell=2}^{\infty} \sum_{\{\sigma, \tau\}} w_{\ell}^0(\sigma_1, \dots, \sigma_{\ell}; \tau_1, \dots, \tau_{\ell}) \prod_{i=1}^{\ell} b_{\sigma_i}^{\dagger} \prod_{j=1}^{\ell} b_{\tau_j}. \quad (29)$$

and similarly for $\hat{\mathcal{V}}_B, \hat{\mathcal{T}}_B$. In the norm operator the ℓ -body terms embody the fact that the fermion-pair operators do not have exactly bosonic commutation relations, and act to enforce the Pauli principle. The coefficients w_ℓ^0 etc. can be written in closed, albeit complicated, form [15].

The norm operator can be conveniently and compactly expressed [17,15,18] in terms of the k th order Casimir operators of the unitary group $SU(2\Omega)$, $\hat{C}_k = 2^k \text{tr} (\mathbf{P})^k$, $\mathbf{P} = \sum_{\sigma\tau} b_\sigma^\dagger b_\tau \mathbf{A}_\sigma \mathbf{A}_\tau^\dagger$ (and so is both a matrix and a boson operator; the trace is over the matrix indices and not the boson Fock space)

$$\hat{\mathcal{N}}_B = : \exp \left(-\frac{1}{2} \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \hat{C}_k \right) : \quad (30)$$

where the colons ‘:’ refer to normal-ordering of the boson operators. This norm operator takes into account the exchange terms in the BZ expansion of a fermion pair given in (13).

Similarly — and this is a new result we have not seen elsewhere in the literature — the representations $\hat{\mathcal{T}}_B, \hat{\mathcal{V}}_B$ can also be written in compact form [15,18]:

$$\hat{\mathcal{T}}_B = 2 \sum_{\sigma,\tau} : \text{tr} [\mathbf{A}_\sigma \mathbf{T} \mathbf{A}_\tau^\dagger \mathbf{G}] b_\sigma^\dagger b_\tau \hat{\mathcal{N}}_B : \quad (31)$$

$$\begin{aligned} \hat{\mathcal{V}}_B = & \sum_{\mu,\nu} \langle \mu | V | \nu \rangle \sum_{\sigma,\tau} : \left\{ \text{tr} [\mathbf{A}_\sigma \mathbf{A}_\mu^\dagger \mathbf{G}] \text{tr} [\mathbf{A}_\nu \mathbf{A}_\tau^\dagger \mathbf{G}] \right. \\ & \left. + 4 \text{tr} [\mathbf{A}_\sigma \mathbf{A}_\mu^\dagger \mathbf{P} \mathbf{G} \mathbf{A}_\mu \mathbf{A}_\tau^\dagger \mathbf{G}] \right\} b_\sigma^\dagger b_\tau \hat{\mathcal{N}}_B : , \end{aligned} \quad (32)$$

where $\mathbf{G} = (\mathbf{1} + 2\mathbf{P})^{-1}$. These compact forms are useful for formal manipulation. Furthermore they have the powerful property of exactly expressing the fermion matrix elements under *any* truncation, a fact not previously appreciated in the literature even for the norm operator [17]. By this we mean the following: suppose we truncate our fermion Fock space to states constructed from a restricted set of pairs $\{\bar{\sigma}\}$. Such a truncation need *not* correspond to any subalgebra. Then the representations in the corresponding truncated boson space, which still exactly reproduce the fermion matrix elements and which we denote by $[\mathcal{N}_B]_T$ etc., are the same as those given above, retaining only the ‘allowed’ bosons with unrenormalized coefficients. For example

$$[\mathcal{N}_B]_T =: \exp \left(-\frac{1}{2} \sum_{k=2}^{\infty} \frac{(-1)^k}{k} [\hat{C}_k]_T \right) : \quad (33)$$

where

$$[\hat{C}_k]_T = 2: \text{tr} ([\mathbf{P}]_T)^k: , \quad [\mathbf{P}]_T = \sum_{\bar{\sigma}\bar{\tau}} b_{\bar{\sigma}}^{\dagger} b_{\bar{\tau}} \mathbf{A}_{\bar{\sigma}} \mathbf{A}_{\bar{\tau}}^{\dagger}. \quad (34)$$

This invariance of the coefficients under truncation will not hold true for the boson *images* introduced below.

With the boson representations of fermion operators in hand, one can express the fermion Schrödinger equation (1) with $\hat{H} = \hat{T} + \hat{V}$ as a generalized boson eigenvalue equation,

$$\hat{\mathcal{H}}_B |\Phi_{\lambda}\rangle = E_{\lambda} \hat{\mathcal{N}}_B |\Phi_{\lambda}\rangle. \quad (35)$$

Here $\hat{\mathcal{H}}_B$ is the boson representation of the fermion Hamiltonian. Every physical fermion eigenstate in (1) has a corresponding eigenstate, with the same eigenvalue, in (35). Because the space of states constructed from pairs of fermions is overcomplete, there also exist spurious boson states that do not correspond to unique physical fermion states. These spurious states will have zero eigenvalues and so can be identified. The overcompleteness also means that (35) is harder to solve exactly than (1). So one truncates the model space.

IV. BOSON IMAGES

In general the boson representations given in (29), (31) and (32) do not have good convergence properties, so that simple termination of the series such as (29) in ℓ -body terms is impossible and use of the generalized eigenvalue equation (35), as written, is problematic. Instead we “divide out” the norm operator to obtain the *boson image*, i.e. schematically,

$$\hat{h} \sim “\hat{\mathcal{H}}_B / \hat{\mathcal{N}}_B.” \quad (36)$$

That this is reasonable is suggested by the explicit forms of (31) and (32). The hope of course is that h is finite or nearly so, so that a 1+2-body fermion Hamiltonian is mapped to an image

$$\hat{h} \sim \theta_1 b^\dagger b + \theta_2 b^\dagger b^\dagger b b + \theta_3 b^\dagger b^\dagger b^\dagger b b b + \theta_4 b^\dagger b^\dagger b^\dagger b^\dagger b b b b + \dots \quad (37)$$

with the ℓ -body terms, $\ell > 2$, zero or greatly suppressed. We now discuss how this “dividing out” is to be carried out.

A. Exact results: Full Space

It turns out that for a number of cases the image of the Hamiltonian is exactly finite. In particular, for the full boson Fock space the representations factor in a simple way: $\hat{T}_B = \hat{\mathcal{N}}_B \hat{T}_B = \hat{T}_B \hat{\mathcal{N}}_B$ and $\hat{V}_B = \hat{\mathcal{N}}_B \hat{V}_B = \hat{V}_B \hat{\mathcal{N}}_B$, where the factored operators \hat{T}_B , \hat{V}_B , which we term the boson images of \hat{T} , \hat{V} , have simple form [15,18]:

$$\hat{T}_B = 2 \sum_{\sigma\tau} \text{tr} \left(\mathbf{A}_\sigma \mathbf{T} \mathbf{A}_\tau^\dagger \right) b_\sigma^\dagger b_\tau, \quad (38)$$

$$\hat{V}_B = \sum_{\mu\nu} \langle \mu | V | \nu \rangle \left[b_\mu^\dagger b_\nu + 2 \sum_{\sigma\sigma'} \sum_{\tau\tau'} \text{tr} \left(\mathbf{A}_\sigma \mathbf{A}_\mu^\dagger \mathbf{A}_{\sigma'} \mathbf{A}_\tau^\dagger \mathbf{A}_\nu \mathbf{A}_{\tau'}^\dagger \right) b_\sigma^\dagger b_{\sigma'}^\dagger b_\tau b_{\tau'} \right] \quad (39)$$

This image Hamiltonian $\hat{H}_B = \hat{T}_B + \hat{V}_B$ is the one determined by BZ if one decomposes the Hamiltonian into multipole-multipole form and then maps these multipole operators. As discussed earlier, these BZ multipole operators are finite in the full space. This result, and its relation to other mappings, was noted by Marshalek [9,6]

Thus any boson representation of a Hamiltonian factorizes: $\hat{\mathcal{H}}_B = \hat{\mathcal{N}}_B \hat{H}_B$ in the full space. Since the norm operator is a function of the $\text{SU}(2\Omega)$ Casimir operators it commutes with the boson images of fermion operators [15,18], and one can simultaneously diagonalize both $\hat{\mathcal{H}}_B$ and $\hat{\mathcal{N}}_B$. Then Eqn. (35) becomes

$$\hat{H}_B |\Phi_\lambda\rangle = E'_\lambda |\Phi_\lambda\rangle. \quad (40)$$

where $E'_\lambda = E_\lambda$ for the physical states, but E'_λ for the spurious states is no longer necessarily zero. The boson Hamiltonian \hat{H}_B is by construction Hermitian and, if one starts with at most only two-body interactions between fermions, has at most two-body boson interactions. All physical eigenstates of the original fermion Hamiltonian will have counterparts in (40).

It should be clear that transition amplitudes between physical eigenstates will be preserved. Spurious states will also exist but, since the norm operator $\hat{\mathcal{N}}_B$ commutes with the boson image Hamiltonian \hat{H}_B , the physical eigenstates and the spurious states will not admix. Also the spurious states can be identified because, while they will no longer have zero energy eigenvalues, they will have eigenvalue zero with respect to the norm operator.

B. Exact Results: Truncated space

The boson Schrödinger equation (40), though finite, is not much use as the boson Fock space is still much larger than the original fermion Fock space, and we still must truncate the boson Fock space. Although the representations remain exact under truncation, the factorization into the image does not persist in general: $[\hat{\mathcal{H}}_B]_T \neq [\hat{\mathcal{N}}_B]_T [\hat{H}_B]_T$. This was recognized by Marshalek [9]. (An alternate formulation [9] does not require the complete Fock space, but mixes physical and spurious states and so always requires a projection operator.)

If the truncation scheme represents a closed subalgebra (specifically, if the truncated set of fermion pairs are closed under double commutations) then a factorization [19]

$$[\hat{\mathcal{H}}_B]_T = [\hat{\mathcal{N}}_B]_T \hat{h}_D = \hat{h}_D^\dagger [\hat{\mathcal{N}}_B]_T \quad (41)$$

does exist, with \hat{h}_D at most two-body, but not necessarily Hermitian. We term it a *Dyson* image [8,6]. Under more restricted conditions on the structure of the pairs and the Hamiltonian one can guarantee \hat{h}_D is Hermitian and commutes with $[\hat{\mathcal{N}}_B]_T$. In the full space, of course, all definitions of boson images coincide and yield the same result.

First, consider a partition of the single fermion states labeled by $i = (i_a, i_c)$, where the dimension of each subspace is $2\Omega_a, 2\Omega_c$ so that $\Omega = 2\Omega_a\Omega_c$. We denote the amplitudes for the truncated space as $\mathbf{A}_\alpha^\dagger$ and assume they can be factored, $(\mathbf{A}_\alpha^\dagger)_{ij} = (\mathbf{K}^\dagger)_{iaja} \otimes (\bar{\mathbf{A}}_\alpha^\dagger)_{icjc}$, with $\mathbf{K}^\dagger \mathbf{K} = \mathbf{K} \mathbf{K}^\dagger = \frac{1}{2\Omega_a}$ and $\mathbf{K}^T = (-1)^p \mathbf{K}$, where $p = 0$ (symmetric) or $p = 1$ (antisymmetric).

Furthermore we assume the completeness relation (8), which was crucial for proving that $\hat{\mathcal{H}}_B = \hat{\mathcal{N}}_B \hat{H}_B$ [15,18], is valid for the truncated space; i.e.,

$$\sum_{\bar{\alpha}} (\bar{\mathbf{A}}_{\bar{\alpha}}^{\dagger})_{i_c j_c} (\bar{\mathbf{A}}_{\bar{\alpha}})_{j'_c i'_c} = \frac{1}{2} \left[\delta_{i_c, i'_c} \delta_{j_c, j'_c} - (-1)^p \delta_{i_c, j'_c} \delta_{i'_c, j_c} \right]. \quad (42)$$

The norm operator in the truncated space then becomes

$$[\hat{\mathcal{N}}_B]_T =: \exp \sum_{k=2} \left(\frac{-1}{\Omega_a} \right)^{k-1} \frac{1}{k} \text{tr}(\bar{\mathbf{P}}^k), \quad (43)$$

where $\bar{\mathbf{P}} = \sum_{\bar{\sigma}\bar{\tau}} b_{\bar{\sigma}}^{\dagger} b_{\bar{\tau}} \bar{\mathbf{A}}_{\bar{\sigma}} \bar{\mathbf{A}}_{\bar{\tau}}^{\dagger}$ so that $[\mathbf{P}]_T = \left(\frac{1}{2\Omega_a} \right) \bar{\mathbf{P}}$. In this case the boson image of a one-body operator is the truncation of the boson image in the full space,

$$[\hat{\mathcal{T}}_B]_T = [\hat{\mathcal{N}}_B]_T [\hat{T}_B]_T \quad (44)$$

$$[\hat{T}_B]_T = 2 \sum_{\bar{\sigma}, \bar{\tau}} \text{tr} \left(\mathbf{A}_{\bar{\sigma}} \mathbf{T} \mathbf{A}_{\bar{\tau}}^{\dagger} \right) b_{\bar{\sigma}}^{\dagger} b_{\bar{\tau}}. \quad (45)$$

The representation of a two-body interaction can be factored into a boson image times the truncated norm,

$$[\hat{\mathcal{V}}_B]_T = [\hat{\mathcal{N}}_B]_T \hat{v}_D; \quad (46)$$

however, \hat{v}_D , while finite (1+2-body), is not simply related to $[\hat{V}_B]_T$ as is the case for one-body operators. If one writes

$$\hat{v}_D = \sum_{\bar{\sigma}, \bar{\tau}} \langle \bar{\sigma} | V | \bar{\tau} \rangle b_{\bar{\sigma}}^{\dagger} b_{\bar{\tau}} + \sum_{\bar{\sigma}\bar{\sigma}'\bar{\tau}\bar{\tau}'} \langle \bar{\sigma}\bar{\sigma}' | v | \bar{\tau}\bar{\tau}' \rangle b_{\bar{\sigma}}^{\dagger} b_{\bar{\sigma}'}^{\dagger} b_{\bar{\tau}} b_{\bar{\tau}'}, \quad (47)$$

then matrix elements of the two-boson interaction are

$$\begin{aligned} \langle \bar{\sigma}\bar{\sigma}' | v | \bar{\tau}\bar{\tau}' \rangle = & \\ & \frac{\sum \langle \mu | V | \nu \rangle}{\Omega_a (2\Omega_a - (-1)^p) (\Omega_a + (-1)^p)} \text{tr}_a \{ \text{tr}_c (\bar{\mathbf{A}}_{\bar{\sigma}} \bar{\mathbf{A}}_{\bar{\tau}}^{\dagger} \mathbf{A}_{\bar{\nu}} \bar{\mathbf{A}}_{\bar{\tau}'}^{\dagger}) \text{tr}_c (\bar{\mathbf{A}}_{\bar{\sigma}'} \mathbf{A}_{\bar{\mu}}^{\dagger}) \\ & + 2\Omega_a [\text{tr}_c (\bar{\mathbf{A}}_{\bar{\sigma}} \mathbf{A}_{\bar{\mu}}^{\dagger} \bar{\mathbf{A}}_{\bar{\sigma}'} \bar{\mathbf{A}}_{\bar{\tau}}^{\dagger} \mathbf{A}_{\bar{\nu}} \bar{\mathbf{A}}_{\bar{\tau}'}^{\dagger}) - \text{tr}_c (\bar{\mathbf{A}}_{\bar{\sigma}} \bar{\mathbf{A}}_{\bar{\mu}}^{\dagger} \bar{\mathbf{A}}_{\bar{\sigma}'} \bar{\mathbf{A}}_{\bar{\tau}}^{\dagger} \mathbf{A}_{\bar{\nu}} \mathbf{K} \bar{\mathbf{A}}_{\bar{\tau}'}^{\dagger})] \\ & - \Omega_a (2\Omega_a + (-1)^p) \text{tr}_c (\mathbf{A}_{\bar{\nu}} \mathbf{K} \bar{\mathbf{A}}_{\bar{\tau}}^{\dagger} \bar{\mathbf{A}}_{\bar{\sigma}'} \bar{\mathbf{A}}_{\bar{\tau}'}^{\dagger}) \delta_{\bar{\sigma}, \mu} \}. \end{aligned} \quad (48)$$

Upon inspection one sees the image is not constrained to be Hermitian.

Consider the additional condition between the matrix elements of the interaction:

$$\begin{aligned}
& \sum_{\mu, \nu} \langle \mu | V | \nu \rangle \sum_{i_a, j_a} (\mathbf{A}_\nu)_{i_a i_c, j_a j_c} (\mathbf{A}_\mu^\dagger)_{j_a j'_c, i_a i'_c} \\
& = N_a \sum_{\mu, \nu} \langle \mu | V | \nu \rangle \sum_{i_a, j_a} (\mathbf{A}_\nu)_{i_a i_c, j_a j_c} (\mathbf{K}^\dagger)_{j_a, i_a} \sum_{i'_a, j'_a} (\mathbf{K})_{i'_a, j'_a} (\mathbf{A}_\mu^\dagger)_{j'_a j'_c, i'_a i'_c}
\end{aligned} \tag{49}$$

where the factor $N_a = \Omega_a(2\Omega_a + (-1)^p)$ is the number of pairs in the excluded subspace. While condition (49) looks complicated there are interactions that satisfy it; for example, two-body interactions constructed from one-body operators $\hat{V} = \hat{T}_{\bar{\alpha}\bar{\beta}} \hat{T}_{\bar{\alpha}'\bar{\beta}'}$ where $\hat{T}_{\bar{\alpha}\bar{\beta}} = [A_{\bar{\alpha}}^\dagger, A_{\bar{\beta}}]$. When (49) is satisfied then \hat{v}_D is Hermitian and although $\hat{v}_D \neq [\hat{V}_B]_T$ they are simply related:

$$\begin{aligned}
\hat{v}_D & = \sum_{\bar{\sigma}, \bar{\tau}} \langle \bar{\sigma} | V | \bar{\tau} \rangle b_{\bar{\sigma}}^\dagger b_{\bar{\tau}} \\
& + 2f_{\Omega_a} \sum_{\mu, \nu} \langle \mu | V | \nu \rangle \sum_{\bar{\sigma}\bar{\sigma}', \bar{\tau}\bar{\tau}'} \text{tr} \left(\mathbf{A}_{\bar{\sigma}} \mathbf{A}_\mu^\dagger \mathbf{A}_{\bar{\sigma}'} \mathbf{A}_\nu^\dagger \mathbf{A}_{\bar{\tau}} \mathbf{A}_{\bar{\tau}'}^\dagger \right) b_{\bar{\sigma}}^\dagger b_{\bar{\sigma}'}^\dagger b_{\bar{\tau}} b_{\bar{\tau}'}
\end{aligned} \tag{50}$$

with $f_{\Omega_a} = 4\Omega_a^2/N_a$ renormalizing the two-boson part of $[\hat{V}_B]_T$ by a factor which ranges from unity (full space) to 2 for a very small subspace. Not all interactions satisfy (49); for example, the pairing interaction never does except in the full space. For the pairing interaction $\langle \mu | V^{\text{pairing}} | \nu \rangle = \delta_{\mu,0} \delta_{\nu,0} G$, and $\mathbf{A}_0 \mathbf{A}_0^\dagger = \frac{1}{2\Omega}$, and the image (46) $\hat{v}_D^{\text{pairing}}$ becomes (remembering $\Omega = 2\Omega_a \Omega_c$)

$$G \left\{ \hat{N}_0 \left[1 - \frac{2}{\Omega} \hat{N} + \frac{1}{\Omega} + \frac{\hat{N}_0}{\Omega} \right] + \sum_{\bar{\tau}\bar{\tau}' \neq 0, \bar{\sigma}} \text{tr} (\bar{\mathbf{A}}_{\bar{\sigma}} \bar{\mathbf{A}}_{\bar{\tau}}^\dagger \bar{\mathbf{A}}_0 \bar{\mathbf{A}}_{\bar{\tau}'}^\dagger) b_{\bar{\sigma}}^\dagger b_0^\dagger b_{\bar{\tau}} b_{\bar{\tau}'} \right\}, \tag{51}$$

where \hat{N} is the total number of bosons, $\hat{N} = \sum_{\bar{\tau}} b_{\bar{\tau}}^\dagger b_{\bar{\tau}}$, and $\hat{N}_0 = b_0^\dagger b_0$. The second term in the above is not Hermitian but can be transformed away by a similarity transformation [20], leaving the first term as a finite Hermitian image which gives the correct eigenvalues for all N .

The $\text{SO}(8)$ and $\text{Sp}(6)$ models [21] belong to a class of models which have a subspace for which (42) is valid and interactions which satisfy (49). In these models the shell model orbitals have a definite angular momentum $\vec{\mathbf{j}}$ and are partitioned into a pseudo orbital angular momentum $\vec{\mathbf{k}}$ and pseudospin $\vec{\mathbf{i}}$, $\vec{\mathbf{j}} = \vec{\mathbf{k}} + \vec{\mathbf{i}}$. The amplitudes are then given as products of Clebsch-Gordon coefficients,

$$(A_{\alpha}^{\dagger})_{ij} = \frac{(1 + (-1)^{K+I})}{2} (k m_i, k m_j | K_{\alpha} M_{\alpha}) (i \mu_i, i \mu_j | I_{\alpha} \mu_{\alpha}), \quad (52)$$

where K and I are the total pseudo orbital angular momentum and pseudospin respectively of the pair of nucleons. For the SO(8) model $\mathbf{i} = \frac{3}{2}$ and one considers the subspace of pairs with $K = 0$ ($p = 0$), $(\bar{A}_{\alpha}^{\dagger})_{ij} = \frac{(1+(-1)^I)}{2} (i \mu_i, i \mu_j | I_{\alpha} \mu_{\alpha})$; in the Sp(6) model $\mathbf{k} = 1$ and one considers the subspace with $I = 0$ ($p = 1$), $(\bar{A}_{\alpha}^{\dagger})_{ij} = \frac{(1+(-1)^K)}{2} (k m_i, k m_j | K_{\alpha} M_{\alpha})$. The complicated conditions (49) hold true for important cases, for example, the quadrupole-quadrupole and other multipole-multipole interactions in the SO(8) and Sp(6) models (that is, interactions of the generic form $P^r \cdot P^r$ in the notation of [21]) have Hermitian Dyson images. Not all interactions in these models have Hermitian Dyson images. For example, pairing in any model (see (51))and, in the SO(8) model, the particular combination $g_0(S^{\dagger}S + \frac{1}{4}P^2 \cdot P^2)$ which is the SO(7) limit. It so happens that these particular cases nonetheless can be brought into finite, Hermitian form as discussed in the next section.

C. Approximate or numerical images

The most general image Hamiltonian one can define is

$$\hat{h} \equiv \mathcal{U} [\tilde{\mathcal{N}}_B]_T^{-1/2} [\hat{\mathcal{H}}_B]_T [\tilde{\mathcal{N}}_B]_T^{-1/2} \mathcal{U}^{\dagger}, \quad (53)$$

which is manifestly Hermitian for any truncation scheme and any interaction, with \mathcal{U} a unitary operator. (Because the norm is a singular operator it cannot be inverted. Instead $[\tilde{\mathcal{N}}_B]_T^{-1/2}$ is calculated from the norm only in the physical subspace, with the zero eigenvalues which annihilate the spurious states retained. Then \hat{h} does not mix physical and spurious states.) If $\mathcal{U} = 1$, this is the democratic mapping [11]. Again, for the full space $\hat{h} = \hat{h}_D = \hat{H}_B$.

This prescription is, we argue, useful for a practical derivation of boson image Hamiltonians. Ignoring for the moment the unitary transformation \mathcal{U} , consider the expansion (37) of \hat{h} . The operators $[\hat{\mathcal{H}}_B]_T$ and $[\tilde{\mathcal{N}}_B]_T^{-1/2}$ have similar expansions, and by multiplying out (53) one sees immediately that the coefficient θ_{ℓ} depends only on up to ℓ -body terms in $[\mathcal{H}_B]_T$ and $[\tilde{\mathcal{N}}_B]_T^{-1/2}$, derived from 2ℓ -fermion matrix elements which are tractable for ℓ

small. Ideally \hat{h} would have at most two-body terms, and our success in finding finite images in the previous section gives us hope that the high-order many-body terms may be small; at any rate the convergence can be calculated and checked term-by-term. Specifically, consider the convergence of the series (37) as a function of ℓ . A rough estimate is that, for an N -boson Fock space, one can truncate to the ℓ -body terms if for $\ell' > \ell$, $\theta_{\ell'}$ is sufficiently small compared to $\theta_{\ell} \times (N - \ell)! / (N - \ell)!$; the strictest condition is to require $\theta_{\ell'} \ll \theta_{\ell} / (\ell' - \ell)!$.

The Hermitian image \hat{h} , defined in (53), is related to the Dyson image \hat{h}_D , defined in (41), by a similarity transformation $\mathcal{S} = \mathcal{U} [\tilde{\mathcal{N}}_B]_T^{1/2}$,

$$\hat{h} = \mathcal{S} \hat{h}_D \mathcal{S}^{-1}. \quad (54)$$

The similarity transformation \mathcal{S} orthogonalizes the fermion states $|\Psi_{\alpha}\rangle$ inasmuch $(\mathcal{S}^{-1})^{\dagger} \tilde{\mathcal{N}}_B \mathcal{S}^{-1} = 1$ in the physical space (and $= 0$ in the spurious space). This is akin to Gram-Schmidt orthogonalization and the freedom to choose \mathcal{U} , and \mathcal{S} , corresponds to the freedom one has in ordering the vectors to be Gram-Schmidt orthogonalized. The OAI and democratic mappings are just two particular choices out of many; the latter takes $\mathcal{U} = 1$. We can use the freedom in the choice of \mathcal{U} to our advantage. Consider the $\text{SO}(8)$ model [21] and its three algebraic limits: the pure pairing interaction, the quadrupole $P^2 \cdot P^2$ interaction, which can be written in terms of $\text{SO}(6)$ Casimir operators, and the linear combination of pairing and quadrupole $S^{\dagger} S + \frac{1}{4} P^2 \cdot P^2$ which can be written in terms of $\text{SO}(7)$ Casimirs (see [21] for details and notation). As discussed in the last Section, the Dyson image of the quadrupole interaction is Hermitian and finite, and $\hat{h}_D = \hat{h}$ with $\mathcal{U} = 1$. The Dyson images of the pairing and $\text{SO}(7)$ interactions are finite but non-Hermitian. We have found \mathcal{U} 's $\neq 1$ for both these cases (but not the same \mathcal{U}) such that their respective Hermitian images \hat{h} are finite; the one for pairing is exactly the OAI prescription, while that for $\text{SO}(7)$ is exactly opposite, orthogonalizing states of low seniority against states of higher seniority. These general Hermitian images do not have a simple relation to the truncated full image, as do the Hermitian Dyson image (50). The one-body piece remains unchanged but there can be significant renormalization, and even change of sign, of the two-body piece. For example, for

the pairing interaction $\left[\hat{H}_B\right]_T = s^\dagger s + \frac{1}{2\Omega^2} s^\dagger s^\dagger s s +$ additional terms, including off-diagonal terms such as $d^\dagger d^\dagger s s$, whereas $\hat{h} = s^\dagger s - \frac{1}{\Omega} s^\dagger s^\dagger s s +$ (depending on the truncation) terms such as $d^\dagger d^\dagger \tilde{d} \tilde{d}$ but *no* off-diagonal terms. Hence we see here the renormalization is not just a simple overall factor f as it was for the Hermitian Dyson image: it is -2Ω for the $s^\dagger s^\dagger s s$ term but 0 for terms such as $d^\dagger d^\dagger s s$.

FIGURES

FIG. 1. Spectrum of SO(7) interaction, for 7 bosons, in SO(8) model with exact (left) and approximate (right) two-body boson Hamiltonians.

If one uses an “inappropriate” transform \mathcal{S} it can induce an unneeded and unwanted many-body dependence. This principle we illustrate in the SO(8) model with the SO(7) interaction, whose spectrum is exactly known and for which we can derive a finite Hermitian image with no many-body dependence; this is the left-hand spectrum in Figure 1. For the right-hand side we took the \mathcal{U} appropriate for pairing, that is the OAI prescription determined for $N = 2$, and calculated the spectrum for $N = 7$ keeping only the strict two-body terms. The distortion in the spectrum from the exact result, such as the overall energy shift and the large perturbation in the third band, indicates missing many-body terms. That is, if one mapped the SO(7) interaction using the canonical OAI procedure one would find of necessity a many-body dependence in the interaction coefficients. By orthogonalizing the basis in the a different way, however, as expressed by a different choice of \mathcal{U} , the many-body dependent vanishes. Therefore it is possible that some of the N -dependence of OAI is

induced by their choice of orthogonalization and could be minimized with a different choice. We are currently exploring how to exploit this freedom to best effect.

V. SUMMARY

In order to investigate rigorous foundations for the phenomenological Interacting Boson Model, we have presented a rigorous microscopic mapping of fermion pairs to bosons, paying special attention to exact mapping of matrix elements, Hermiticity, truncation of the model space, and many-body terms. First we presented new, general and compact forms for boson *representations* that preserve fermion matrix elements. We then considered the boson *image* Hamiltonian which results from “dividing out” the norm from the representation; in the full boson Fock space the image is always finite and Hermitian; in addition we discussed several analytic cases for truncated spaces where the image is also finite and Hermitian. Finally, we give a prescription which is a generalization of both the OAI and democratic mappings; in the most general case for truncated spaces the Hermitian image Hamiltonian may not be finite but we have demonstrated there is some freedom in the mapping that one could possibly exploit to minimize the many-body terms. This freedom, which manifests itself in a similarity transformation that orders the orthogonalization of the underlying fermion basis, depends on the Hamiltonian.

This research was supported by the U.S. Department of Energy. The boson calculations for Figure 1 were performed using the PHINT package of Scholten [22].

REFERENCES

- [1] S.T. Belyaev and V.G. Zelevinskii, *Nucl. Phys.* **39** (1962) 582.
- [2] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, 1987).
- [3] T. Otsuka, A. Arima, F. Iachello and I. Talmi, *Phys. Lett.* **76B** (1978) 139.
- [4] T. Otsuka, A. Arima, and F. Iachello, *Nucl. Phys.* **A309** (1978) 1.
- [5] I. Talmi, *Simple Models of Complex Nuclei*, (Harwood Academic Publishers, 1994).
- [6] A. Klein and E. R. Marshalek, *Rev. Mod. Phys.* **63** (1991) 375; P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer-Verlag, 1980).
- [7] T. Marumori, M. Yamamura, and A. Tokunaga, *Prog. Theor. Phys.* **31** (1964) 1009; T. Marumori, M. Yamamura, A. Tokunaga, and A. Takeda, *Prog. Theor. Phys.* **32** (1964) 726.
- [8] F.J. Dyson *Phys. Rev.* **102** (1956) 1217.
- [9] E. R. Marshalek, *Phys. Rev. C* **38** (1988) 2961.
- [10] T. Kishimoto and T. Tamura, *Phys. Rev. C* **27** (1983) 341; H. Sakamoto and T. Kishimoto, *Nucl. Phys.* **A486** (1988) 1.
- [11] L. D. Skouras, P. van Isacker, and M. A. Nagarajan, *Nucl. Phys.* **A516** (1990) 255.
- [12] B. Silvestre-Brac and R. Piepenbring *Phys. Rev. C* **26** (1982) 2640.
- [13] D.J. Rowe, T. Song and H. Chen, *Phys. Rev. C* **44**(1991) R598.
- [14] G.H. Lang, C.W. Johnson, S.E. Koonin, and W.E. Ormand, *Phys. Rev. C* **48** (1993) 1518.
- [15] J.N. Ginocchio and C.W. Johnson, to be published.

- [16] T. Otsuka and N. Yoshinaga, *Phys. Lett.* **168B** (1986) 1.
- [17] J. Dobaczewski, H.B. Geyer, and F.J.W. Hahne, *Phys. Rev. C* **44** (1991) 1030.
- [18] C.W. Johnson and J.N. Ginocchio, in press, *Phys. Rev. C*, August 1994.
- [19] J.N. Ginocchio and C.W. Johnson, to be published in “Frontiers of Nuclear Structure Physics”, T. Otsuka, ed. (World Scientific, 1994).
- [20] J.N. Ginocchio and I. Talmi, *Nucl. Phys.* **A337** (1980) 431.
- [21] J.N. Ginocchio, *Ann. Phys.* **126** (1980) 234.
- [22] O. Scholten in *Computational Nuclear Physics 1*, K. Langanke, J.A. Maruhn and S.E. Koonin, eds., (Springer-Verlag 1991) Chapter 5.

This figure "fig1-1.png" is available in "png" format from:

<http://arXiv.org/ps/nucl-th/9408014v1>